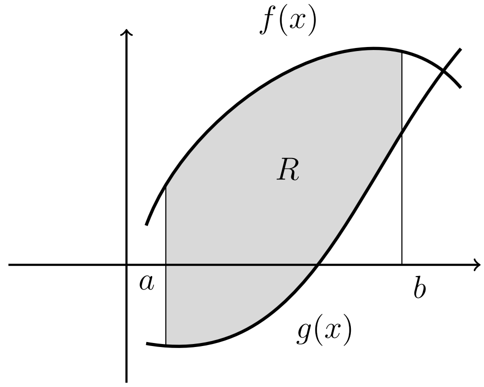
Objectives:

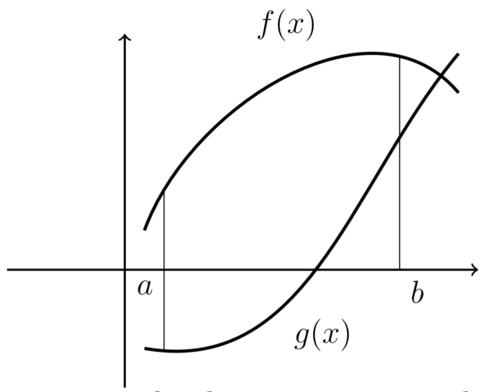
- Compute the areas between curves.

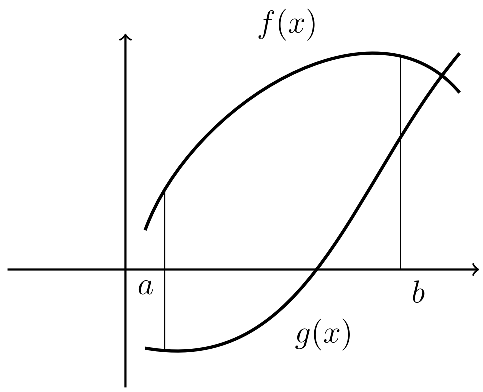
Motivation: We've been computing the between a curve and the x-axis. How can we compute the area between two curves?

Consider the region, R, between the two curves $y = f(x)$ and $y = g(x)$ between the vertical lines $x = a$ and $x = b$ where f and g are continuous functions and $f(x) \geq g(x)$ for all x in $[a, b]$.



What if we used rectangles?





How can we make this approximation better?

Increase n, just like before! In fact, this approximation will get better and better as $n \to \infty$.

We can express this area, A, in terms of a Riemann sum:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]\Delta x$$

But wait! This limit is the definite integral of $f - g$.

Definition: The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$, and the lines $x = a$ and $x = b$ where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\, dx$$

Example 1 Find the area of the region bounded by the curves $f(x) = x^3$ and $g(x) = 0$ between $x = 1$ and $x = 5$.

On the interval $[1, 5]$, $f(x) = x^3$ is always greater than $g(x) = 0$, so the upper curve is $f(x)$.

$$\int_1^5 x^3 - 0\ dx = \frac{x^4}{4}\Big|_1^5 = \frac{5^4}{4} - \frac{1}{4} = \frac{624}{4} = 156$$

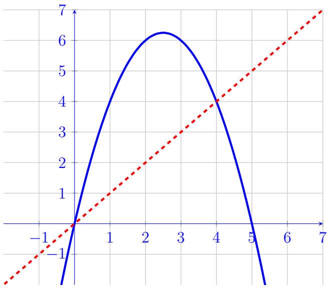
Example 2 Find the area of the region bounded by the curves $f(x) = e^x$ and $g(x) = x$ between $x = 0$ and $x = 1$.

The upper curve on $[0, 1]$ is $f(x) = e^x$. (Draw a graph to convince yourself.)

$$\int_0^1 e^x - x\ dx = \left(e^x - \frac{x^2}{2}\right)\Big|_0^1 = \left(e - \frac{1}{2}\right) - (1 - 0) = e - \frac{3}{2}$$

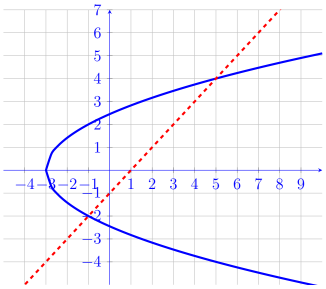
Example 3 Find the area of the region enclosed by the parabola $y = 5x - x^2$ and the line $y = x$.

First, we need to find out at what x-values the enclosed region starts and ends.



The intersections of the two curves are whenever $x = 5x - x^2$. Solving for x, we find the intersections are at $x = 0, 4$. We also note that the upper curve is $y = 5x - x^2$. So, the area is given by $\int_0^4 (5x - x^2) - x\ dx = \int_0^4 4x - x^2\ dx = \left(2x^2 - \frac{x^3}{3}\right)\Big|_0^4 = 2(16) - \frac{64}{3} - (0 - 0) = \frac{32}{3}$

Example 4 Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



Since $y^2 = 2x + 6$ is not a function of x we need to either (1) split up the integral into a ton of smaller pieces or (2) integrate with respect to y. It's going to be a lot easier to integrate with respect to y.
The first step then is to rewrite each equation as a function of y, so we have: $x = y + 1$ and $x = \frac{1}{2}y^2 - 3$.
The intersections of these curves are where $y + 1 = \frac{1}{2}y^2 - 3$, which occurs at $y = 4, -2$.
The "upper curve" in this case is the curve with larger x-values rather than the curve with larger y-values. So the upper curve is $x = y + 1$.
So! The area is given by

$$\int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right)\ dx = \left(y^2 + y - \frac{1}{6}y^3 - 3y\right)\Big|_{-2}^{4} = -12$$